Toeplitz-Hausdorff like theorem for matrices over quaternions

Santhosh Kumar Pamula

Indian Statistical Institute Bangalore

August 23, 2019

Otto Toeplitz

Felix Hausdorff

1918: Otto Teplitz, *Das algebraische Analogon zu einem Satze von Fejér*, Math. Z.

1919: Felix Hausdorff, *Der Wertvorrat einer Bilinearform*, Math. Z.

Quaternions

- $\mathbb{H}$: Skew-field of Hamilton quaternions.
- An element $q \in \mathbb{H}$ is of the form $q = q_0 + q_1 i + q_2 j + q_3 k$, where i, j, k are fundamental quaternion units satisfying:

$$i^2 = j^2 = k^2 = -1 = ijk.$$

- $Re(q) = q_0,\ Im(q) = q_1 i + q_2 j + q_3 k$ and $\overline{q} = Re(q) - Im(q)$.
- The modulus of q is, $|q| = \sqrt{q_0^2 + q_1^2 + q_2^2 + q_3^2}$ and the imaginary unit sphere is, $\mathbb{S} = \{q \in \mathbb{H} :\ Re(q) = 0, |q| = 1\}$.
- For each $m \in \mathbb{S}$, the slice $\mathbb{C}_m := \{a + bm :\ a, b \in \mathbb{R}\} \cong \mathbb{C}$.
- If $q \in \mathbb{H}$, then $q = q_0 + m_q |Im(q)|$, where $m_q = \frac{Im(q)}{|Im(q)|} \in \mathbb{S}$.

Quaternions

- For $p, q \in \mathbb{H}$, define $p \sim q$ if and only if $p = s^{-1}qs$, for some $s \in \mathbb{H} \setminus \{0\}$.
- It is an equivalence relation on $\mathbb{H}$ and the equivalence class,

$$[q] = \left\{p \in \mathbb{H} : \ Re(p) = Re(q), \ |Im(p)| = |Im(q)|\right\}.$$

Note that $[q] \cap \mathbb{C} = \left\{Re(q) \pm i|Im(q)|\right\}$ for every $q \in \mathbb{H}$.

Quaternions

- For $p, q \in \mathbb{H}$, define $p \sim q$ if and only if $p = s^{-1}qs$, for some $s \in \mathbb{H} \setminus \{0\}$.
- It is an equivalence relation on $\mathbb{H}$ and the equivalence class,

$$[q] = \{p \in \mathbb{H} : \ Re(p) = Re(q), \ |Im(p)| = |Im(q)|\}.$$

Note that $[q] \cap \mathbb{C} = \{Re(q) \pm i|Im(q)|\}$ for every $q \in \mathbb{H}$.

Definition

1. A subset $\mathcal{K} \subset \mathbb{H}$ is said to be *circular* or *axially symmetric* if $[q] \in \mathcal{K}$ for all $q \in \mathcal{K}$.
2. For $\mathbb{V} \subseteq \mathbb{C}$, the *circularization* $\Omega_{\mathbb{V}}$ is defined by

$$\Omega_{\mathbb{V}} := \{a + mb : \ a + ib \in \mathbb{V}, m \in \mathbb{S}\}.$$

Quaternionic numerical range

- $\mathbb{H}^n$ is a right $\mathbb{H}$-module and the innerproduct is given by,

$$\big\langle (x_i), (y_i) \big\rangle_{\mathbb{H}} = \sum_{i=1}^{n} \overline{x_i} y_i, \ \forall \ (x_i), (y_i) \in \mathbb{H}^n.$$

- The unit sphere in $\mathbb{H}^n$ is, $S_{\mathbb{H}^n} = \big\{ X \in \mathbb{H}^n : \ \|X\| = 1 \big\}$.

Quaternionic numerical range

- $\mathbb{H}^n$ is a right $\mathbb{H}$-module and the innerproduct is given by,

$$\big\langle (x_i), (y_i) \big\rangle_{\mathbb{H}} = \sum_{i=1}^{n} \overline{x_i} y_i, \ \forall \, (x_i), (y_i) \in \mathbb{H}^n.$$

- The unit sphere in $\mathbb{H}^n$ is, $S_{\mathbb{H}^n} = \big\{ X \in \mathbb{H}^n : \ \|X\| = 1 \big\}$.

Definition

The quaternionic numerical range of $A \in M_n(\mathbb{H})$ is defined by

$$\mathrm{W}_{\mathbb{H}}(A) = \big\{ \langle X, AX \rangle_{\mathbb{H}} : \ X \in S_{\mathbb{H}^n} \big\}.$$

It is a compact and circular subset of $\mathbb{H}$.

Is $W_{\mathbb{H}}(A)$ convex?

Example:

$$\text{Let } A = \begin{bmatrix} k & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{3\times 3} \in M_3(\mathbb{H}).$$

Then $k, -k \in W_{\mathbb{H}}(A)$, but $0 \notin W_{\mathbb{H}}(A)$.

Is $W_{\mathbb{H}}(A)$ convex?

Example:

$$\text{Let } A = \begin{bmatrix} k & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{3\times 3} \in M_3(\mathbb{H}).$$

Then $k, -k \in W_{\mathbb{H}}(A)$, but $0 \notin W_{\mathbb{H}}(A)$.

To see this: Suppose $0 = \langle X, AX \rangle_{\mathbb{H}}$ for $X = (x_1, x_2, x_3) \in S_{\mathbb{H}^3}$, then

$$\overline{x_1} k x_1 + |x_2|^2 + |x_3|^2 = 0.$$

This is a contradiction, since $Re(\overline{x_1} k x_1) = 0$.

So, the quaternionic numerical range is not necessarily convex.

History

1936: L.A. Wolf, *Similarity of matrices in which the elements are real quaternions*, Bull. Amer. Math. Soc.

History

1936: L.A. Wolf, *Similarity of matrices in which the elements are real quaternions*, Bull. Amer. Math. Soc.

1949: H. C. Lee, *Eigenvalues and canonical forms of matrices with quaternion coefficients*, Proc. Roy. Irish Acad.

1951: J. L. Brenner, *Matrices of quaternions*, Pacific J. Math.

1936: L.A. Wolf, *Similarity of matrices in which the elements are real quaternions*, Bull. Amer. Math. Soc.

1949: H. C. Lee, *Eigenvalues and canonical forms of matrices with quaternion coefficients*, Proc. Roy. Irish Acad.

1951: J. L. Brenner, *Matrices of quaternions*, Pacific J. Math.

- The study of the convexity of $W_{\mathbb{H}}(A)$ as a subset of $\mathbb{H}$ has begun by Kippenhahn and later followed by Wiegmann.

1951: R. von Kippenhahn, *Über den Wertevorrat einer Matrix*, Math. Nachr.

1955: N. A. Wiegmann, *Some theorems on matrices with real quaternion elements*, Canad. J. Math.

History

- J.E. Jamison proposed a problem to characterize the class of linear operators on quaternionc Hilbert space with convex numerical range.

1972: J.E. Jamison, *Numerical Range and Numerical Radius in Quaternionic Hilbert spaces*, Doctoral Dissertation, Univ. of Missouri.

- Propoerties of $W_{\mathbb{H}}(A) \cap \mathbb{R}$ and $W_{\mathbb{H}}(A) \cap \mathbb{C}$ are well studied.

1984: Au-Yeung, *On the convexity of numerical range in quaternionic Hilbert spaces*, Linear Multilinear Alg.

History

1993: F. Zhang, *Permanant Inequalities and Quaternion matrices*, Ph.D. Dissertataion, Univ. of California at Santa Barbara.

1994: W. So, R. C. Thompson and F. Zhang, *Numerical ranges of matrices with quaternion entries*, Linear and Multilinear Alg.

1995: F. Zhang, *On Numerical Range of Normal matrices of Quaternions*, J. Math. Physical Sciences.

History

1993: F. Zhang, *Permanant Inequalities and Quaternion matrices*, Ph.D. Dissertataion, Univ. of California at Santa Barbara.

1994: W. So, R. C. Thompson and F. Zhang, *Numerical ranges of matrices with quaternion entries*, Linear and Multilinear Alg.

1995: F. Zhang, *On Numerical Range of Normal matrices of Quaternions*, J. Math. Physical Sciences.

- So and Thompson gave a proof (65 pages long).

1996: W. So and R.C. Thompson, *Convexity fo the upper complex plane part of the numerical range of a quternion matrix*, Linear Multilinear Alg.

History

- In 1997, Zhang posed three questions.

Question 1 : Is there a short and conceptual proof to show that $W_{\mathbb{H}}(A) \cap \mathbb{C}^+$ is convex ?

Question 2 : How is $W_{\mathbb{H}}(A) \cap \mathbb{C}$ related to corresponding complex matrix ?

Question 3: Investigate $W_{\mathbb{H}}(A)$ and $W_{\mathbb{H}}(A) \cap \mathbb{C}^+$ when A is bounded linear operator on infinite dimensional right quaternionic Hilbert space?

1997: F. Zhang, *Quaternions and matrices of quaternions,* Linear algebra Appl.

Relation with complex matrices

Definition

Let $A \in M_n(\mathbb{H})$. Then

1. for every $m \in \mathbb{S}$, $W_{\mathbb{H}}(A) \cap \mathbb{C}_m^+$ is called $\mathbb{C}_m$-section of $W_{\mathbb{H}}(A)$. In particular,

$$W_{\mathbb{H}}^+(A) := W_{\mathbb{H}}(A) \cap \mathbb{C}^+.$$

2. $W_{\mathbb{H}}(A : \mathbb{C}) := \left\{ co(q) : \ q \in W_{\mathbb{H}}(A) \right\}$, where $co(q) = q_0 + q_1 i$.

Relation with complex matrices

Definition

Let $A \in M_n(\mathbb{H})$. Then

1. for every $m \in \mathbb{S}$, $W_{\mathbb{H}}(A) \cap \mathbb{C}_m^+$ is called $\mathbb{C}_m$-section of $W_{\mathbb{H}}(A)$. In particular,

$$W_{\mathbb{H}}^+(A) := W_{\mathbb{H}}(A) \cap \mathbb{C}^+.$$

2. $W_{\mathbb{H}}(A : \mathbb{C}) := \left\{ co(q) : \ q \in W_{\mathbb{H}}(A) \right\}$, where $co(q) = q_0 + q_1 i$.

Note that if $A \in M_n(\mathbb{H})$,then $A = A_1 + A_2\, j$, for $A_1, A_2 \in M_n(\mathbb{C})$. Define

$$\chi_A = \begin{bmatrix} A_1 & A_2 \\ -\overline{A_2} & \overline{A_1} \end{bmatrix}_{2n \times 2n} \in M_{2n}(\mathbb{C}).$$

Relation with complex matrices

Theorem (S., 2019)

Let $A \in M_n(\mathbb{H})$. Then $W_{\mathbb{H}}(A : \mathbb{C}) = W_{\mathbb{C}}(\chi_A)$.

Relation with complex matrices

Theorem (S., 2019)

Let $A \in M_n(\mathbb{H})$. Then $W_{\mathbb{H}}(A : \mathbb{C}) = W_{\mathbb{C}}(\chi_A)$.

- This mplies that $W_{\mathbb{H}}(A) \subseteq \Omega_{W_{\mathbb{C}}(\chi_A)}$. The equality may not hold.

Relation with complex matrices

Theorem (S., 2019)
Let $A \in M_n(\mathbb{H})$. Then $W_{\mathbb{H}}(A : \mathbb{C}) = W_{\mathbb{C}}(\chi_A)$.

- This mplies that $W_{\mathbb{H}}(A) \subseteq \Omega_{W_{\mathbb{C}}(\chi_A)}$. The equality may not hold.

Example: Let $A = j \in \mathbb{H}$, then $\chi_A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \in M_2(\mathbb{C})$ and

$$\left\langle \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} \right\rangle_{\mathbb{H}} = 0.$$

That is, $0 \in \Omega_{W_{\mathbb{C}}(\chi_A)}$, but $0 \notin W_{\mathbb{H}}(A)$ since $j \in \mathbb{S}$.

Connectedness properties

Theorem (Au-Yeung, 1984)

Let $A \in M_n(\mathbb{H})$. Then

1. for any $\alpha \in \mathbb{R}$, the set $\{X \in S_{\mathbb{H}^n} : \langle X, AX \rangle_{\mathbb{H}} = \alpha\}$ is connected if $A = A^*$
2. the set $\{X \in S_{\mathbb{H}^n} : \langle X, AX \rangle_{\mathbb{H}} = 0\}$ is connected if $A = -A^*$.

Theorem (Au-Yeung, 1984)

Let $A \in M_n(\mathbb{H})$. Then

1. for any $\alpha \in \mathbb{R}$, the set $\{X \in S_{\mathbb{H}^n} : \langle X, AX \rangle_{\mathbb{H}} = \alpha\}$ is connected if $A = A^*$
2. the set $\{X \in S_{\mathbb{H}^n} : \langle X, AX \rangle_{\mathbb{H}} = 0\}$ is connected if $A = -A^*$.

Corollary

Let $A \in M_n(\mathbb{H})$. Then $W_{\mathbb{H}}(A) \cap \mathbb{R}$ is either *empty set* or *connected*.

Proof

Since $A = \frac{1}{2}(A + A^*) + \frac{1}{2}(A - A^*)$, we see that

$$W_{\mathbb{H}}(A) \cap \mathbb{R} = \left\{X \in S_{\mathbb{H}^n} : \langle X, (A - A^*)X \rangle_{\mathbb{H}} = 0\right\}.$$

From above Theorem, It follows that $W_{\mathbb{H}}(A) \cap \mathbb{R}$ is connected.

Connectedness properties

Lemma (S., 2019)

Let $A \in M_n(\mathbb{H})$ and let L be any line parallel to Y-axis. Then $W^+_{\mathbb{H}}(A) \cap L$ is connected.

Proposition (S., 2019)

Let $\mathbb{V}$ be a finite subset of $\mathbb{C}$. Then

$$Conv\left(\Omega_{\mathbb{V}}\right) = Conv\left(\Omega_{Conv(\mathbb{V})}\right).$$

Here $Conv(\cdot)$ is an abbreviation for 'Convex hull of'.

Result for $M_2(\mathbb{H})$

Lemma (S., 2019)

Let $A \in M_2(\mathbb{H})$. Then every section of $W_{\mathbb{H}}(A)$ is convex.

Result for $M_2(\mathbb{H})$

Lemma (S., 2019)

Let $A \in M_2(\mathbb{H})$. Then every section of $W_{\mathbb{H}}(A)$ is convex.

proof

By the canonical form of [Brenner, 1951] there exist a unitary $U \in M_2(\mathbb{H})$ such that

$$A = U^* \begin{bmatrix} z_1 & p \\ 0 & z_2 \end{bmatrix} U,$$

for some $p \in \mathbb{H}$ and $z_1, z_2 \in \mathbb{C}^+$.

Result for $M_2(\mathbb{H})$

Lemma (S., 2019)

Let $A \in M_2(\mathbb{H})$. Then every section of $W_{\mathbb{H}}(A)$ is convex.

proof

By the canonical form of [Brenner, 1951] there exist a unitary $U \in M_2(\mathbb{H})$ such that

$$A = U^* \begin{bmatrix} z_1 & p \\ 0 & z_2 \end{bmatrix} U,$$

for some $p \in \mathbb{H}$ and $z_1, z_2 \in \mathbb{C}^+$. Now we show that the quaternionic numerical range of $\begin{bmatrix} z_1 & p \\ 0 & z_2 \end{bmatrix}$ is convex. Let $\begin{bmatrix} x \\ y \end{bmatrix} \in S_{\mathbb{H}^2}$. Then consider the following cases.

Result for $M_2(\mathbb{H})$

Case(1): $z_1 = z_2 = z := a + ib, \ p = 0$

Result for $M_2(\mathbb{H})$

Case(1): $z_1 = z_2 = z := a + ib, \; p = 0$

$$\left\langle \begin{bmatrix} x \\ y \end{bmatrix}, \begin{bmatrix} a+ib & o \\ 0 & a+ib \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \right\rangle_{\mathbb{H}} = a(|x|^2 + |y|^2) + bm_{x,y},$$

where $m_{x,y} = \overline{x}ix + \overline{y}iy$.

Result for $M_2(\mathbb{H})$

Case(1): $z_1 = z_2 = z := a + ib,\ p = 0$

$$\left\langle \begin{bmatrix} x \\ y \end{bmatrix}, \begin{bmatrix} a+ib & o \\ 0 & a+ib \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \right\rangle_{\mathbb{H}} = a(|x|^2 + |y|^2) + b m_{x,y},$$

where $m_{x,y} = \overline{x} i x + \overline{y} i y$. Clearly, $Re(m_{x,y}) = 0$ and $|m_{x,y}| \leq 1$. That is,

$$\left\{ m_{x,y} : |x|^2 + |y|^2 = 1 \right\} \subseteq \left\{ q \in \mathbb{H} : Re(q) = 0,\ |q| \leq 1 \right\}.$$

Result for $M_2(\mathbb{H})$

Case(1): $z_1 = z_2 = z := a + ib,\ p = 0$

$$\left\langle \begin{bmatrix} x \\ y \end{bmatrix}, \begin{bmatrix} a+ib & o \\ 0 & a+ib \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \right\rangle_{\mathbb{H}} = a(|x|^2 + |y|^2) + b m_{x,y},$$

where $m_{x,y} = \overline{x} i x + \overline{y} i y$. Clearly, $Re(m_{x,y}) = 0$ and $|m_{x,y}| \leq 1$. That is,

$$\left\{ m_{x,y} : |x|^2 + |y|^2 = 1 \right\} \subseteq \left\{ q \in \mathbb{H} : Re(q) = 0,\ |q| \leq 1 \right\}.$$

If $q \in \mathbb{H} \setminus \{0\}$ such that $Re(q) = 0$ and $|q| \leq 1$, then $\exists\ s \neq 0$ with $s^{-1} i s = \frac{q}{|q|}$. Take

$$x = \sqrt{\frac{1+|q|}{2}} \cdot \frac{s}{|s|},\quad y = \sqrt{\frac{1-|q|}{2}} \cdot \frac{s}{|s|}$$

Result for $M_2(\mathbb{H})$

Then $|x|^2 + |y|^2 = 1$ and $m_{x,y} = q$.

Result for $M_2(\mathbb{H})$

Then $|x|^2 + |y|^2 = 1$ and $m_{x,y} = q$.
If $q = 0$, then by choosing $x = \frac{1}{\sqrt{2}}, y = j\frac{1}{\sqrt{2}}$ we get $m_{x,y} = 0$.
This shows that

$$\{m_{x,y} : |x|^2 + |y|^2 = 1\} = \{q \in \mathbb{H} : Re(q) = 0,\ |q| \leq 1\}.$$

Therefore,

$$W_{\mathbb{H}}(A) = \{a + bm :\ Re(m) = 0 \text{ with } 0 \leq |m| \leq 1\}.$$

Result for $M_2(\mathbb{H})$

Then $|x|^2 + |y|^2 = 1$ and $m_{x,y} = q$.
If $q = 0$, then by choosing $x = \frac{1}{\sqrt{2}}, y = j\frac{1}{\sqrt{2}}$ we get $m_{x,y} = 0$.
This shows that

$$\{m_{x,y} : |x|^2 + |y|^2 = 1\} = \{q \in \mathbb{H} : Re(q) = 0,\ |q| \leq 1\}.$$

Therefore,

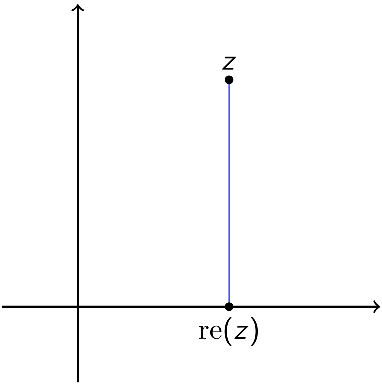
$$W_{\mathbb{H}}(A) = \{a + bm : \ Re(m) = 0 \text{ with } 0 \leq |m| \leq 1\}.$$

It is the solid sphere in $\mathbb{R}^4$ with radius b and center at $(a, 0, 0, 0)$. So $W_{\mathbb{H}}(A)$ is convex.

In particular, $W_{\mathbb{H}}^{+}(A)$ is the line segment joining $Re(z)$ and z, which is convex.

Result for $M_2(\mathbb{H})$

Graph of $W_{\mathbb{H}}^{+}(A)$:



Result for $M_2(\mathbb{H})$

Case(2): $z_1 = a_1 + ib_1, z_2 = a_2 + ib_2, p = 0$

Result for $M_2(\mathbb{H})$

Case(2): $z_1 = a_1 + ib_1, z_2 = a_2 + ib_2, p = 0$

$$\left\langle \begin{bmatrix} x \\ y \end{bmatrix}, \begin{bmatrix} a_1 + ib_1 & o \\ 0 & a_2 + ib_2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \right\rangle_{\mathbb{H}} = a_1|x|^2 + a_2|y|^2 + b_1\overline{x}ix + b_2\overline{y}iy.$$

Result for $M_2(\mathbb{H})$

Case(2): $z_1 = a_1 + ib_1, z_2 = a_2 + ib_2, p = 0$

$$\left\langle \begin{bmatrix} x \\ y \end{bmatrix}, \begin{bmatrix} a_1 + ib_1 & o \\ 0 & a_2 + ib_2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \right\rangle_{\mathbb{H}} = a_1|x|^2 + a_2|y|^2 + b_1\overline{x}ix + b_2\overline{y}iy.$$

Suppose its imaginary part is zero, i.e.,

$$b_1\overline{x}ix = -b_2\overline{y}iy. \tag{1}$$

Since $|x|^2 + |y|^2 = 1$, we get

$$|x| = \sqrt{\frac{b_2}{b_1 + b_2}}, \quad |y| = \sqrt{\frac{b_1}{b_1 + b_2}}. \tag{2}$$

From Equations (1), (2), we get

$$x^{-1}ix + y^{-1}iy = 0. \tag{3}$$

Result for $M_2(\mathbb{H})$

In fact, Equation (1) $\Leftrightarrow$ Equations (2) & (3).

Result for $M_2(\mathbb{H})$

In fact, Equation (1) $\Leftrightarrow$ Equations (2) & (3). The only choice is,

$$x = \sqrt{\frac{b_2}{b_1 + b_2}},\ y = \sqrt{\frac{b_1}{b_1 + b_2}}.$$

Therefore,

$$W_{\mathbb{H}}(A) \cap \mathbb{R} = \left\{ v := \frac{a_1 b_2 + a_2 b_1}{b_1 + b_2} \right\}.$$

Result for $M_2(\mathbb{H})$

In fact, Equation (1) $\Leftrightarrow$ Equations (2) & (3). The only choice is,

$$x = \sqrt{\frac{b_2}{b_1 + b_2}}, \; y = \sqrt{\frac{b_1}{b_1 + b_2}}.$$

Therefore,

$$W_{\mathbb{H}}(A) \cap \mathbb{R} = \left\{ v := \frac{a_1 b_2 + a_2 b_1}{b_1 + b_2} \right\}.$$

Claim: $W_{\mathbb{H}}^{+}(A) = Conv\big(\{z_1, z_2, v\}\big)$.
In particular, if we take $x, y \in \mathbb{C}$ with $|x^2 + |y|^2 = 1$, then
$z_1|x|^2 + z_2|y|^2 \in W_{\mathbb{H}}^{+}(A)$.

Result for $M_2(\mathbb{H})$

We show that the line segment joining v and z_1 is in $W_H^+(A)$. Let $u_t := a_1(1-t) + vt$, $x_t = \sqrt{\frac{a_2-u_t}{a_2-a_1}}$ and $y_t = j\sqrt{\frac{u_t-a_1}{a_2-a_1}}$ for $t \in [0,1]$. Then $|x_t|^2 + |y_t|^2 = 1$ with

$$\left\langle \begin{bmatrix} x_t \\ y_t \end{bmatrix}, \begin{bmatrix} a_1 + ib_1 & 0 \\ 0 & a_2 + ib_2 \end{bmatrix} \begin{bmatrix} x_t \\ y_t \end{bmatrix} \right\rangle_{\mathbb{H}} = (a_1 + ib_1)(1-t) + vt.$$

Similarly, the line joining v and z_2 is in $W_H^+(A)$.

Result for $M_2(\mathbb{H})$

We show that the line segment joining v and z_1 is in $W_H^+(A)$. Let $u_t := a_1(1-t)+vt$, $x_t = \sqrt{\frac{a_2-u_t}{a_2-a_1}}$ and $y_t = j\sqrt{\frac{u_t-a_1}{a_2-a_1}}$ for $t \in [0,1]$. Then $|x_t|^2 + |y_t|^2 = 1$ with

$$\left\langle \begin{bmatrix} x_t \\ y_t \end{bmatrix}, \begin{bmatrix} a_1+ib_1 & 0 \\ 0 & a_2+ib_2 \end{bmatrix} \begin{bmatrix} x_t \\ y_t \end{bmatrix} \right\rangle_{\mathbb{H}} = (a_1+ib_1)(1-t)+vt.$$

Similarly, the line joining v and z_2 is in $W_H^+(A)$. By the fact that $W_{\mathbb{H}}^+(A) \cap L$ is connected, we get that

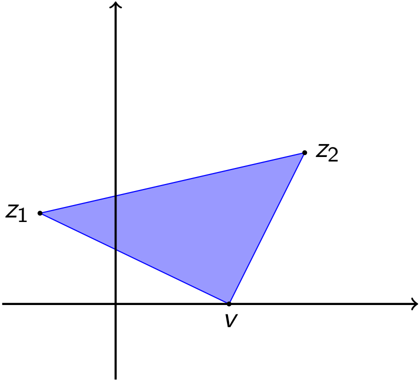
$$Conv(\{z_1, z_2, v\}) \subseteq W_{\mathbb{H}}^+(A).$$

Finally, the equality holds since

$$W_{\mathbb{H}}^+(A) \subseteq Conv(\Omega_{\{z_1,z_2,v\}}) = Conv(\Omega_{Conv(\{z_1,z_2,v\})}).$$

Result for $M_2(\mathbb{H})$

Graph of $W_{\mathbb{H}}^{+}(A)$:



Result for $M_2(\mathbb{H})$

Case(3): $z_1 = z_2 = 0$.

Result for $M_2(\mathbb{H})$

Case(3): $z_1 = z_2 = 0$.
By Young's Inequality, we have

$$\begin{aligned} \left| \left\langle \begin{bmatrix} x \\ y \end{bmatrix}, \begin{bmatrix} 0 & p \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \right\rangle_{\mathbb{H}} \right| &= |\overline{x} p y| \\ &\leq |p| \cdot \frac{|x|^2 + |y|^2}{2} \\ &= \frac{|p|}{2}. \end{aligned}$$

Result for $M_2(\mathbb{H})$

Case(3): $z_1 = z_2 = 0$.
By Young's Inequality, we have

$$\left| \left\langle \begin{bmatrix} x \\ y \end{bmatrix}, \begin{bmatrix} 0 & p \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \right\rangle_{\mathbb{H}} \right| = \left| \overline{x} p y \right| \leq |p| \cdot \frac{|x|^2 + |y|^2}{2} = \frac{|p|}{2}.$$

Let $|p| = 1$. Then for any q with $|q| \leq \frac{1}{2}$, we have $q = re^{m_q \theta}, 0 \leq r \leq \frac{1}{2}$ where $m_q = \frac{Im(q)}{|Im(q)|}$. If we choose $x = e^{-m_q \theta} cos\alpha$ and $y = p^{-1} sin\alpha$ such that $sin\, 2\alpha = 2r \leq 1$ and $0 \leq \alpha \leq \frac{\pi}{4}$, then $\overline{x} p y = q$.

Result for $M_2(\mathbb{H})$

It shows that $W_{\mathbb{H}}(A) = \{q \in \mathbb{H} : \ |q| \leq \frac{1}{2}\}$. If $|p| \neq 1$, then we have

$$W_{\mathbb{H}}(A) = W_{\mathbb{H}}\left(\begin{bmatrix} 0 & \frac{p}{|p|} \\ 0 & 0 \end{bmatrix} \right)|p| = \left\{q \in \mathbb{H} : \ |q| \leq \frac{|p|}{2}\right\}.$$

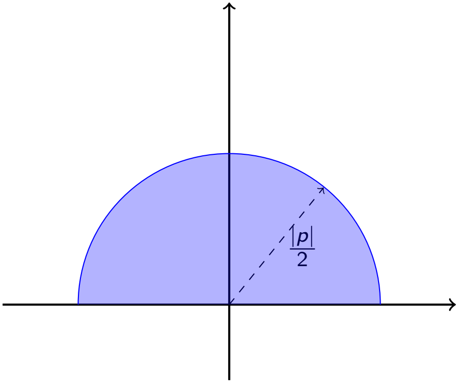
Therefore,

$$W_{\mathbb{H}}^{+}(A) = \left\{z \in \mathbb{C}^{+} : \ |z| \leq \frac{|p|}{2}\right\}.$$

It is the upper half of the disc with radius $\frac{|p|}{2}$.

Result for $M_2(\mathbb{H})$

Graph of $W_{\mathbb{H}}^{+}(A)$:



Result for $M_2(\mathbb{H})$

Case(4): $z_1 = a_1 + ib_1, z_2 = a_2 + ib_2, p \neq 0$

Since $\Gamma := \left\{ u + \tau : \ u \in W^+_{\mathbb{H}}(\begin{bmatrix} z_1 & 0 \\ 0 & z_2 \end{bmatrix}), \ \tau \in \mathbb{C}^+ \text{ with } \ |\tau| \leq \frac{|p|}{2} \right\}$ is convex and $W^+_{\mathbb{H}}(A) \cap L$ is connected, it shows that $W^+_{\mathbb{H}}(A)$ is convex.

Graph of $W^+_{\mathbb{H}}(A)$: It is clear that for any $\lambda \in W^+_{\mathbb{H}}(A)$, we have

$$\lambda = \overline{x} z_1 x + \overline{y} z_2 y + \overline{x} p y, \text{ for some } \begin{bmatrix} x \\ y \end{bmatrix} \in S_{\mathbb{H}^2}$$

and $|\lambda| \leq \max\{|z_1|, |z_2|\} + \frac{|p|}{2}$.

Therefore, $W^+_{\mathbb{H}}(A)$ is a convex subset of upper half of the disc with radius $R := \max\{|z_1|, |z_2|\} + \frac{|p|}{2}$.

Result for $M_2(\mathbb{H})$

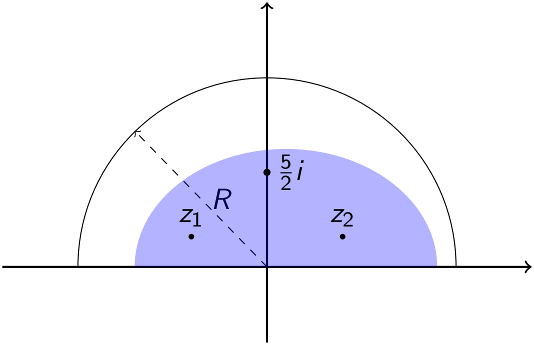
There is no guarantee that either $Re(p) + |Im(p)|i$ or $\frac{|p|}{2}i$ lies in $W_{\mathbb{H}}^{+}(A)$. The following are the examples of three different possibilities.

Result for $M_2(\mathbb{H})$

There is no guarantee that either $Re(p) + |Im(p)|i$ or $\frac{|p|}{2}i$ lies in $W_{\mathbb{H}}^{+}(A)$. The following are the examples of three different possibilities.

Example 1

If $z_1 = -1 + i, z_2 = 1 + i$ and $p = 3 - 4k$, then $3 + 4i \notin W_{\mathbb{H}}^{+}(A)$, but $\frac{|p|}{2}i = \frac{5}{2}i \in W_{\mathbb{H}}^{+}(A)$.



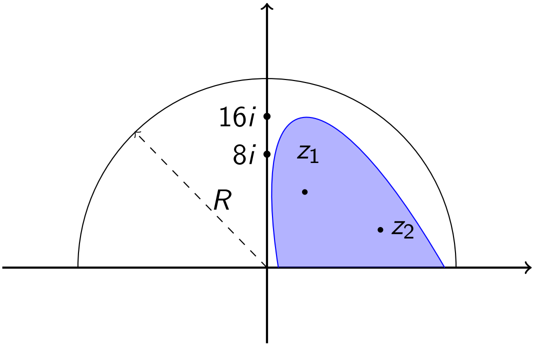
Result for $M_2(\mathbb{H})$

Example 2

If $z_1 = 3 + 4i, z_2 = 20 + i, p = 16j$, then

$$\text{neither } re(p) + |im(p)|i = 16i \text{ nor } \frac{|p|}{2}i = 8i \text{ lies in } W^+_{\mathbb{H}}(A).$$

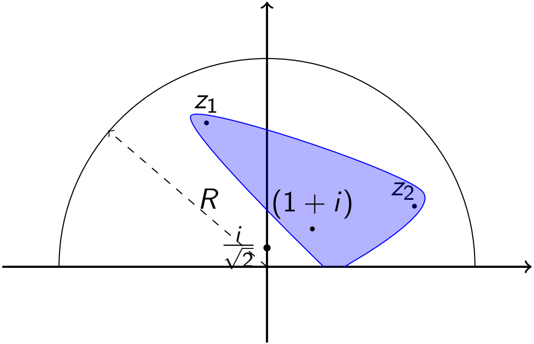
.



Example 3

Let $z_1 = 3 + 4i, z_2 = -2 + 5i$ and $p = 1 - j$, then

$$re(p) + |im(p)|i = 1 + i \in W^+_{\mathbb{H}}(A), \text{ but } \frac{|p|}{2} i = \frac{i}{\sqrt{2}} \notin W^+_{\mathbb{H}}(A).$$



Toeplitz-Hausdorff like theorem

Theorem (S., 2019)

Let $A \in M_n(\mathbb{H})$. Then every section of $W_{\mathbb{H}}(A)$ is convex.

Proof

Suppose $z, w \in W_{\mathbb{H}}^{+}(A)$, then

$$z = \langle X, AX \rangle_{\mathbb{H}}, \quad w = \langle Y, AY \rangle_{\mathbb{H}}$$

for some $X, Y \in S_{\mathbb{H}^n}$. We show that the line segment joining z and w contained in $W_{\mathbb{H}}^{+}(A)$. Let V be the two-dimensional subspace generated by z, w, which is isomorphic to $\mathbb{H}^2$ and let P be the projection of $\mathbb{H}^2$ onto V.

Then $PAP\big|_V \in M_2(\mathbb{H})$ with

$$\langle X, PAPX\rangle_{\mathbb{H}} = z,\ \langle Y, PAPY\rangle_{\mathbb{H}} = w.$$

This shows that $z, w \in W^+_{\mathbb{H}}(PAP\big|_V)$. Since $W^+_{\mathbb{H}}(PAP\big|_V)$ is convex (from previous Lemma), the line segment joining z and w is contined in $W^+_{\mathbb{H}}(PAP\big|_V) \subseteq W^+_{\mathbb{H}}(A)$.

Hence $W^+_{\mathbb{H}}(A)$ is convex.

References

P. Santhosh Kumar, *A note on convexity of sections of quaternionic numerical range*, Linear Algebra Appl. 572 (2019), 92-116.

K.E. Gustafson and D.K.M. Rao, *Numerical Range*, Universitext, Springer-Verlag, New York (1997).

W. So, R.C. Thompson and F. Zhang, *The numerical range of normal matrices with quaternion entries*, Linear Multilinear Algebra 37 (1994).

F. Zhang, *Quaternions and matrices of quaternions*, Linear Algebra Appl. 251 (1997) 21-57.

THANK YOU